INSTABILITY OF MONOCHROMATIC WAVES ON THE SURFACE OF A LIQUID OF ARBITRARY DEPTH

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A study is made of the stability, with respect to the spontaneous appearance of modulation, of steady periodic waves of small amplitude on the surface of an ideal liquid depth. The well-known instability of waves on the surface of a liquid of infinite depth disappears on making the transition to small depths.

As is well known, nonlinear steady waves on the surface of a liquid are unstable [1, 4]. The instability increment describing the rate of growth of waves on the surface of a liquid of infinite depth is obtained in its most general form in [1]. The case of infinite depth is analyzed in detail in [2], where inter alia capillary effects are taken into account. The instability of waves on the surface of a liquid of finite depth is examined in [3, 4], although the analysis is restricted to the one-dimensional problem where the wave vectors of the perturbations are parallel to the wave vector of the initial wave. In the present article we discuss the instability of waves on the surface of a liquid of finite depth for an arbitrary direction of perturbation wave vector. We assume, however, that the wave vectors of the perturbations are sufficiently close to the wave vector of the initial wave that the instability can be represented as a spontaneous growth of modulation on a background of the initial wave. Such "modulation" instabilities are known for waves in a nonlinear dielectric [5].

We consider the potential flow of an ideal liquid of arbitrary depth in a uniform gravitational field. The coordinate system is chosen so that, in its unperturbed state, the surface of the liquid lies in the xy plane. The x axis points out of the liquid. All vector quantities relate to two-dimensional vectors in the xy plane.

Let $\eta(\mathbf{r}, t)$ denote the shape of the liquid surface and $\Phi(\mathbf{r}, z, t)$ the hydrodynamic potential. The flow of the liquid is described by Laplace's equation with two boundary conditions at the surface and one at the bottom:

$$\Delta\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1}$$

$$\frac{\partial\eta}{\partial t} = \sqrt{1 + (\nabla\eta)^2}\, \frac{\partial\Phi}{\partial n}\Big|_{z=\eta} = \frac{\partial\Phi}{\partial z}\Big|_{z=\eta} - \nabla\eta\nabla\Phi\Big|_{z=\eta} \tag{2}$$

$$\frac{\partial\Phi}{\partial t} + g\eta = -\frac{(\nabla\Phi)^2}{2}\Big|_{z=\eta} - \frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)^2\Big|_{z=\eta} \tag{3}$$

$$\frac{\partial\Phi}{\partial z}\Big|_{z=-h} = 0 \tag{4}$$

The total energy of the liquid is given by

$$E = \frac{1}{2}\int d\mathbf{r} \int_{-h}^{\eta} \left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] dz + \frac{g}{2}\int \eta^2 d\mathbf{r} \tag{5}$$

We define

$$\Psi(\mathbf{r}, t) = \Phi(\mathbf{r}, z, t)|_{z=\eta}$$

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By virtue of the uniqueness of the solution of the boundary problem for Laplace's equation, the flow of the liquid is completely determined when η and Ψ are assigned.

Utilizing the formula

$$\frac{\partial \Psi}{\partial T} = \frac{\partial \Phi}{\partial t}\Big|_{z=\eta} + \frac{\partial \eta}{\partial t}\frac{\partial \Phi}{\partial z}\Big|_{z=\eta}$$

we obtain from Eq. (3)

$$\frac{\partial \Psi}{\partial t} + g\eta = -\frac{1}{2}(\nabla\Phi)^2\Big|_{z=\eta} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2\Big|_{z=\eta} - \frac{\partial \Phi}{\partial z}(\nabla\eta\nabla\Phi)\Big|_{z=\eta} \tag{6}$$

As shown in [2], η and Ψ are canonical variables, the energy E is the Hamiltonian of the system, and Eqs. (2), (6) are Hamilton's equations.

We shall now proceed to solve the boundary problem for Laplace's equation. First of all, we carry out a Fourier transformation with respect to the coordinates x and y

$$\eta(\mathbf{k}) = \frac{1}{2\pi}\int \eta(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}, \quad \Psi(\mathbf{k}) = \frac{1}{2\pi}\int \Psi(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}$$

The general solution of Laplace's equation in which the boundary condition at the bottom has been taken into account is given by

$$\Phi(\mathbf{k}, z) = A(\mathbf{k})e^{|\mathbf{k}|z}[1 + e^{-z|\mathbf{k}|(z+h)}] \tag{7}$$

In the subsequent discussion we shall seek a solution in the form of a series in powers of η, restricting ourselves to terms of order not higher than η^2. Remembering that

$$\Phi(\mathbf{k}, z)|_{z=\eta} = \Psi(\mathbf{k})$$

and expanding the exponential as a series, we obtain

$$\begin{aligned}\Phi(\mathbf{k}, z) = \frac{\operatorname{ch}|\mathbf{k}|(z+h)}{\operatorname{ch}|\mathbf{k}|h}\Big\{\Psi(\mathbf{k}) + \int \Psi(\mathbf{k}_1)\eta(\mathbf{k}_2)|\mathbf{k}_1|\operatorname{th}|\mathbf{k}_1|h\delta(\mathbf{k}\\ - \mathbf{k}_1 - \mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2 - \frac{1}{2}\int [|\mathbf{k}-\mathbf{k}_3|\operatorname{th}|\mathbf{k}-\mathbf{k}_3|h + |\mathbf{k}-\mathbf{k}_2|\operatorname{th}|\mathbf{k}-\mathbf{k}_2|h\\ - |\mathbf{k}|\operatorname{th}|\mathbf{k}|h]|\mathbf{k}_1|\Psi(\mathbf{k}_1)\eta(\mathbf{k}_2)\eta(\mathbf{k}_3)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3\Big\}\end{aligned} \tag{8}$$

Performing a canonical transformation to the complex-variable $a(\mathbf{k})$-amplitudes of travelling waves through the formulas

$$\begin{aligned}\eta(\mathbf{k}) &= \frac{1}{\sqrt{2}}\left(\frac{|\mathbf{k}|\operatorname{th}|\mathbf{k}|h}{\omega(\mathbf{k})}\right)^{1/2}[a(\mathbf{k}) + \bar{a}(-\mathbf{k})]\\ \Psi(\mathbf{k}) &= -\frac{1}{\sqrt{2}}\left(\frac{\omega(\mathbf{k})}{|\mathbf{k}|\operatorname{th}|\mathbf{k}|h}\right)^{1/2}[a(\mathbf{k}) - \bar{a}(-\mathbf{k})]\end{aligned} \tag{9}$$

we obtain the Hamiltonian in the form

$$\begin{aligned}H = \int \omega(\mathbf{k})a(\mathbf{k})\bar{a}(\mathbf{k})d\mathbf{k} + \int V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)[\bar{a}(\mathbf{k})a(\mathbf{k}_1)a(\mathbf{k}_2) + a(\mathbf{k})\bar{a}(\mathbf{k}_1)\\ \times \bar{a}(\mathbf{k}_2)]\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2 + {}^1/_3\int U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)[a(\mathbf{k})a(\mathbf{k}_1)a(\mathbf{k}_2)\\ + \bar{a}(\mathbf{k})\bar{a}(\mathbf{k}_1)\bar{a}(\mathbf{k}_2)]\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2 + {}^1/_2\int W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\\ \times \bar{a}(\mathbf{k})\bar{a}(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3\end{aligned} \tag{10}$$

Here $\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|\operatorname{th}|\mathbf{k}|h}$ is the dispersion law for gravitational waves. The expressions for the functions $V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$, $U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ and $W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ are cumbersome and will not be reproduced here.

The equations of motion are obtained by variation of the Hamiltonian in accordance with the laws

$$\frac{\partial a(\mathbf{k})}{\partial t} = -i\frac{\delta H}{\delta \bar{a}(\mathbf{k})}, \quad \frac{\partial \bar{a}(\mathbf{k})}{\partial t} = i\frac{\delta H}{\delta a(\mathbf{k})} \tag{11}$$

This gives for $a(\mathbf{k})$

$$\frac{\partial a(\mathbf{k})}{\partial t} + i\omega(\mathbf{k})a(\mathbf{k}) = -i\int\{V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)a(\mathbf{k}_1)a(\mathbf{k}_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$

$$+2V(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2)\, a(\mathbf{k}_2)\, \bar{a}(\mathbf{k}_1)\, \delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2) + U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, \bar{a}(\mathbf{k}_1)\, \bar{a}(\mathbf{k}_2)$$
$$\times \delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\}\, d\mathbf{k}_1 d\mathbf{k}_2 - i\int W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, \bar{a}(\mathbf{k}_1)\, a(\mathbf{k}_2)\, a(\mathbf{k}_3) \tag{12}$$
$$\times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

Let the period of the steady wave be $L = 2\pi/k_0$. Its main harmonic is then $A\delta(\mathbf{k}-\mathbf{k}_0)$. Nonlinear interaction leads to the appearance of zeroth and second harmonics. Assuming that the wave is weakly modulated as a result of instability, we write the function $a(\mathbf{k})$ in the form

$$a(\mathbf{k}) = b(\mathbf{k}) + a_1(\mathbf{k}) + a_2(\mathbf{k}) \tag{13}$$

where $b(\mathbf{k})$, $a_1(\mathbf{k})$, $a_2(\mathbf{k})$ are concentrated respectively near $\mathbf{k} = 0$, $\mathbf{k} = \mathbf{k}_0$ and $\mathbf{k} = 2\mathbf{k}_0$.

The following conditions hold for the weak nonlinearity under investigation:

$$a_2(\mathbf{k}) \ll a_1(\mathbf{k}), \qquad b(\mathbf{k}) \ll a_1(\mathbf{k})$$

This gives for $a_2(\mathbf{k})$ when (13) is inserted into equation of motion (12):

$$\frac{\partial a_2(\mathbf{k})}{\partial t} + i\omega(\mathbf{k})\, a_2(\mathbf{k}) = -i\int V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}-\mathbf{k}_1)\, a_1(\mathbf{k})\, a_1(\mathbf{k}-\mathbf{k}_1)\, d\mathbf{k}_1 \tag{14}$$

or, assuming a sufficiently narrow wave packet,

$$a_2(\mathbf{k}) = -\frac{V(2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)} \int a_1(\mathbf{k}_1)\, a_1(\mathbf{k}-\mathbf{k}_1)\, d\mathbf{k}_1 \tag{15}$$

Let us insert (13) into Hamiltonian (10), expressing $a_2(\mathbf{k})$ by formula (15). Further, we shall retain in the Hamiltonian only those terms which contain $a_1(\mathbf{k})$ in the form of the product $a_1(\mathbf{k})a_1(\mathbf{k})$, since the contribution from other terms will be small.

The resulting simplified Hamiltonian has the form

$$H = \int \omega(\mathbf{k})\, a(\mathbf{k})\, \bar{a}(\mathbf{k})\, d\mathbf{k} + \int \omega(\mathbf{k})\, b(\mathbf{k})\, \bar{b}(\mathbf{k})\, d\mathbf{k} + \int f(\mathbf{k})\, [b(\mathbf{k})\, a(\mathbf{k}_1)\, \bar{a}(\mathbf{k}_2)$$
$$+ \bar{b}(\mathbf{k})\, \bar{a}(\mathbf{k}_1)\, a(\mathbf{k}_2)]\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 + \frac{\lambda}{2}\int \bar{a}(\mathbf{k})\, \bar{a}(\mathbf{k}_1)\, a(\mathbf{k}_2)\, a(\mathbf{k}_3) \tag{16}$$
$$\times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

where we have dropped the index 1 in $a_1(\mathbf{k})$. Here

$$\lambda = W(2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0) - 2\frac{V^2(2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)} - 2\frac{U^2(-2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)}{\omega(2\mathbf{k}_0) + 2\omega(\mathbf{k}_0)} \tag{17}$$

$$f(\mathbf{k}) = 2V(\mathbf{k}, \mathbf{k}_0, \mathbf{k}_0) \tag{18}$$

Hamiltonian (16) corresponds to the equations of motion

$$\frac{\partial a(\mathbf{k})}{\partial t} + i\omega(\mathbf{k})\, a(\mathbf{k}) = -i\int [f(\mathbf{k}_1)\, b(\mathbf{k}_1) + f(-\mathbf{k}_1)\, \bar{b}(-\mathbf{k}_1)]\, a(\mathbf{k}_2)\, \delta(\mathbf{k} \tag{19}$$
$$-\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2 - i\lambda\int \bar{a}(\mathbf{k}_1)\, a(\mathbf{k}_2)\, a(\mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

$$\frac{\partial b(\mathbf{k})}{\partial t} + i\omega(\mathbf{k})\, b(\mathbf{k}) = -if(\mathbf{k})\int \bar{a}(\mathbf{k}_1)\, a(\mathbf{k}_2)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2 \tag{20}$$

If $f(0) = 0$, system (19), (20) has the exact solution

$$b(\mathbf{k}) = 0, \quad a_0(\mathbf{k}) = Ae^{-it[\omega(\mathbf{k}_0)+\lambda|A|^2]}\, \delta(\mathbf{k}-\mathbf{k}_0) \tag{21}$$

which represents a monochromatic wave with a frequency which depends on its amplitude.

We now investigate the stability of solution (21) with respect to small perturbations of the form

$$a(\mathbf{k}) = a_0(\mathbf{k}) + \alpha(\mathbf{k})\, e^{-i\omega t - i\Omega t}\delta(\mathbf{k}-\mathbf{k}_0-\varkappa) + \alpha(\mathbf{k})\, e^{-i\omega t + i\Omega t}\, \delta(\mathbf{k}-\mathbf{k}_0+\varkappa) \tag{22}$$
$$b(\mathbf{k}) = \beta(\mathbf{k})\, e^{-i\Omega t} + \beta(\mathbf{k})\, e^{i\Omega t}$$

Any perturbation can be represented as a superposition of perturbations (22).

The following fourth-order dispersion equation is obtained on inserting (22) into the equations of motion (9) and (20):

$$
\begin{aligned}
&(\Omega^2-\omega^2(\varkappa))(-\Omega+\omega(\mathbf{k}_0+\varkappa)-\omega(\mathbf{k}_0-\varkappa))\\
&\times(\Omega+\omega(\mathbf{k}_0-\varkappa)-\omega(\mathbf{k}_0))+A^2(\omega(\mathbf{k}_0+\varkappa)+\omega(\mathbf{k}_0-\varkappa)-2\omega(\mathbf{k}_0))\\
&\times[f^2(\varkappa)(\Omega+\omega(\varkappa))+f^2(-\varkappa)(\omega(\varkappa)-\Omega)+\lambda(\Omega^2-\omega^2(\varkappa))]=0
\end{aligned}
\tag{23}
$$

Since the spectrum of gravitational waves does not collapse [6], one would expect an instability to develop when the wave vectors satisfy (see [1])

$$\omega(\mathbf{k}_0+\varkappa)+\omega(\mathbf{k}_0-\varkappa)=2\omega(\mathbf{k}_0) \tag{24}$$

The wave vectors lie near the surface:

$$\omega(\mathbf{k}_0+\varkappa)-\omega(\mathbf{k}_0)=\omega(\mathbf{k}_0)-\omega(\mathbf{k}_0-\varkappa)=\Omega_0$$

In the case under investigation $|\varkappa| \ll k_0$ we can put $\Omega = \Omega_0 + \delta\Omega$. We then have, correct to second order terms in $\delta\Omega$,

$$(\delta\Omega)^2-(L\varkappa^2)^2-2A^2\varkappa^2LG=0 \tag{25}$$

Here

$$G=\lambda-\frac{f^2(\varkappa)}{\omega(\varkappa)-\Omega_0}-\frac{f^2(-\varkappa)}{\omega(\varkappa)+\Omega_0} \tag{26}$$

$$L=\frac{1}{2}k_\alpha k_\beta\frac{\partial^2\omega(\mathbf{k})}{\partial k_\alpha\partial k_\beta} \tag{27}$$

The instability increment is then given by

$$\gamma=\sqrt{2A^2\varkappa^2LG+(L\varkappa^2)^2} \tag{28}$$

In this manner, we have a condition for stability at small $|\varkappa|$, namely, $LG > 0$.

The functions L and G have the following form in the present case:

$$
\begin{aligned}
L&=\frac{\sqrt{gh}}{2k_0\sqrt{x\,\mathrm{th}\,x}}\left\{\left[\frac{2x}{\mathrm{ch}^2x}(1-x\,\mathrm{th}\,x)-\frac{1}{2\,\mathrm{th}\,x}\left(\mathrm{th}\,x+\frac{x}{\mathrm{ch}^2x}\right)^2\right]\cos^2\theta+\left(\mathrm{th}\,x+\frac{x}{\mathrm{ch}^2x}\right)\sin^2\theta\right\}\\
G&=-\frac{k_0^3}{32\pi^2}\left\{\left[\sqrt{\frac{x}{\mathrm{th}\,x}}(1+\mathrm{th}^2x)-2\cos\theta\right]^2\frac{1}{x(1-\alpha\cos\theta)}\right.\\
&+\left[\sqrt{\frac{x}{\mathrm{th}\,x}}(1+\mathrm{th}^2x)+2\cos\theta\right]^2\frac{1}{x(1+\alpha\cos\theta)}+4(\mathrm{th}\,x-2\,\mathrm{th}\,2x\,\mathrm{th}\,x)\\
&\left.+\frac{1}{2}\left[\sqrt{\frac{2\,\mathrm{th}\,2x}{\mathrm{th}\,x}}(1+\mathrm{th}^2x)+4(\mathrm{th}\,x\,\mathrm{th}\,2x-1)\right]^2\frac{1}{\mathrm{th}\,2x-2\,\mathrm{th}\,x}\right\}
\end{aligned}
\tag{29}
$$

Here

$$\alpha=\frac{1}{2\sqrt{x\,\mathrm{th}\,x}}\left(\mathrm{th}\,x+\frac{x}{\mathrm{ch}^2x}\right),\qquad x=k_0h \tag{30}$$

If x is not too small,

$$
\begin{aligned}
L&\approx {}^1\!/_2\,k_0^{-1}\sqrt{gh}\sqrt{x^{-1}\,\mathrm{th}\,x}\,(1-{}^3\!/_2\cos^2\theta)\\
G&>0
\end{aligned}
\tag{31}
$$

Evidently, the instability increment is at its greatest when $\theta = 0$, and the instability develops for angles less than a critical value.

For $x \ll 1$

$$
\begin{aligned}
G&\approx\frac{k_0^3}{32\pi^2x^3}\left\{1-\frac{x^2}{1-\cos\theta+{}^1\!/_2x^2\cos\theta}\right\}\\
L&\approx\sqrt{\frac{gh}{k_0}}(\sin^2\theta-x^2\cos^2\theta)
\end{aligned}
\tag{32}
$$

In this case, evidently, the instability cannot develop at large angles. For small angles

$$G \sim (\theta^2 - x^2),\ L \sim (\theta^2 - x^2)$$

i.e., $LG \sim (\theta^2 - x^2)$ is always greater than zero. In this manner, in the first order in x, waves on the surface of a liquid of small depth are stable. Investigation of the stability in the next order in x requires a more exact theory.

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